UNIVERSAL TYPE STRUCTURES WITH UNAWARENESS

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Abstract. Infinite hierarchies of awareness and beliefs arise in games with unawareness, similarly to belief hierarchies in standard games. A natural question is whether each hierarchy describes the player's awareness of the hierarchies of other players and beliefs over these, or whether the reasoning can continue indefinitely. This paper constructs the universal type structure with unawareness where each type has an awareness level and a belief over types. Countable hierarchies are therefore sufficient to describe all uncertainty in games with unawareness.

JEL classification: C72; D82; D83

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1. Introduction

There are many situations where the payoff of an agent depends on the actions of other agents and on uncertain external factors. The actions of the agents depend on their reasoning, so to choose the best action, an agent has to reason about the reasoning of others, their reasoning about the reasoning of others, and so on to arbitrarily high order. With complete information, this infinite regress can be avoided by imposing a fixed-point equilibrium concept, but with incomplete information, the process generates infinite hierarchies of reasoning. A natural question is whether the infinite hierarchies summarize all the uncertainty in the game or whether it is necessary to go even further, describing a player's reasoning about the opponents' infinite hierarchies, about their reasoning about their opponents' hierarchies, etc.

In standard games, the reasoning of an agent is described by a probability distribution over known outcomes (exogenous uncertainties and other agents' possible beliefs). The seminal paper of Mertens and Zamir (1985) was the first to show that the initial hierarchies of reasoning are sufficient—each hierarchy encodes a probability distribution over the set of hierarchies, thus closing the model.

Not all uncertainty is describable by a probability distribution, but infinite regress is a feature of any kind of interactive reasoning. The same question of closure of the hierachical model then arises as for probabilistic beliefs. If the uncertainty is described by conditional probability systems, compact continuous possibility correspondences or compact sets of probabilities, the papers of Battigalli and Siniscalchi (1999); Mariotti, Meier, and Piccione (2005) and Ahn (2007) show that the hierarchies capture all uncertainty about the hierarchies.

On the other hand, for knowledge and similar information structures, reasoning about other agents' reasoning may continue indefinitely, as shown in Heifetz and Samet (1998a); Meier (2005), therefore there may be no level of the hierarchies that fully describes all uncertainty in the model. With additional assumptions on the knowledge operators, closure can be obtained (Meier, 2008; Mariotti, Meier, and Piccione, 2005).

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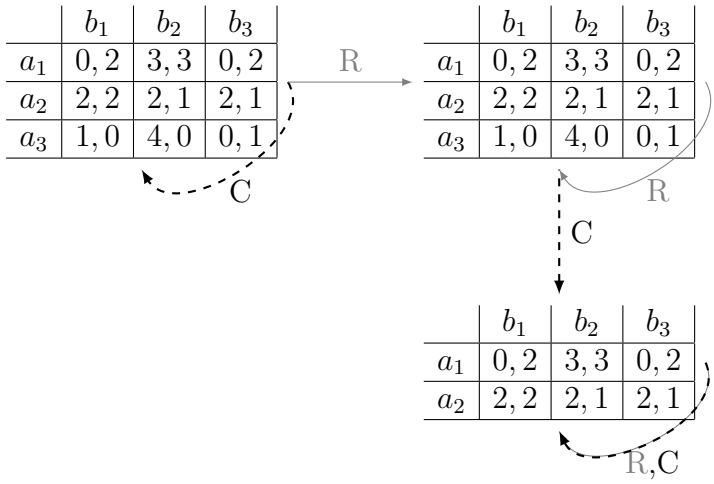

FIGURE 1. A game with unawareness from Feinberg (2009).

All the abovementioned kinds of uncertainty share the property that the agents know all possible outcomes. In many real-world decision problems this is not the case—the agents may be unaware of some aspects of the environment. An easily noticeable difference between unawareness and zero probability is that unawareness is symmetric—if an agent is unaware of an event, he is unaware of its negation. Zero probability on an event implies putting probability one on its negation. Choice under unawareness may exhibit 'reverse Bayesianism', i.e. updating may lead an initially null event to receive positive weight in decisions (Karni and Vierø, 2010). The experiment of Mengel, Tsakas, and Vostroknutov (2011) shows that being exposed to unawareness increases the risk aversion of experimental subjects.

Unawareness may give rise to novel behaviour, as illustrated in the game in Fig. 1, which consists of three normal form games, with beliefs (represented by arrows) between them. At the top left normal form, the column player C believes the game to be the full Fig. 1, while R believes the game to be restricted to the two normal forms on the right side. At the top right normal form, R again believes the game to be restricted to the two normal forms on the right side, while C believes the game to consist of just the bottom right normal form (a standard game). At the bottom right normal form, both believe the game to be the bottom right. So at the top left game, C believes that R believes that C believes the bottom right normal form is the whole game. All beliefs put probability one on a single game in this example, but this need not be the case in other games with unawareness.

The solution of the game in Fig. 1 is reminiscent of backward induction. In the bottom right normal form, there are two pure-strategy equilibria: (a_2, b_1) and (a_1, b_2). Focus on the payoff-dominant (a_1, b_2). In the game consisting of the two right-side normal forms, R believes C imagines himself in the bottom right normal form and plays b_2. Then R best responds with a_3. In the whole game, C believes R imagines herself to be in the game consisting of the two right-side normal forms and plays a_3, so C best responds with b_3. The equilibrium (a_3, b_3) of the game with unawareness is absent from both the standard game consisting of just the top left normal form in Fig. 1 and from the standard game consisting of just the bottom right normal form. The addition of unawareness has changed the equilibrium set.

To better describe such situations, unawareness has been added to games by Grant and Quiggin (2009); Halpern (2008); Rêgo and Halpern (2012); Heifetz, Meier, and Schipper (2011b). In games with unawareness, the players are aware of only some aspects of the game, form beliefs about external uncertainties and other players' awareness and beliefs, and so on, giving rise to infinite

hierarchies of awareness and belief. Similarly to standard games, there is a question as to whether the hierarchies encapsulate all uncertainty in the model. Given the results in the literature that the hierarchical model closes for some kinds of uncertainty, but not for others, the answer is not obvious in the case of unawareness.

This paper proves that infinite hierarchies of awareness and belief include the description of awareness and belief about the hierarchies if agents are unaware of aspects of the space of primitive uncertainty, other agents or higher-order reasoning.

In propositional unawareness, the space of primitive uncertainty is the set of states of nature. Awareness levels are based on partitions of the space of primitive uncertainty. An agent with a particular awareness level can reason about the partition corresponding to that level, about others' reasoning about that partition, their reasoning about his reasoning about the partition, etc. Propositional unawareness is the most common kind used in the literature, in both propositional models (Halpern, 2001; Heifetz, Meier, and Schipper, 2008; Halpern and Rêgo, 2008) and set-based ones (Modica and Rustichini, 1999; Heifetz, Meier, and Schipper, 2006; Li, 2009). Unawareness of available actions, as in Feinberg's game in Fig. 1 and in Heifetz, Meier, and Schipper (2007, 2011a), can be described using the same mathematical framework as for propositional unawareness.[1] This is made possible by using the space of primitive uncertainty to describe uncertainty about the availability of actions, as well as payoff uncertainty. It should be emphasized that despite the mathematical similarity, there is still a conceptual difference between unawareness of actions and propositional unawareness.

An agent who is unaware of other agents does not include all other agents in his subjective model. Awareness levels are based on sets of agents, so an agent with limited awareness is able to reason only about a subset of agents, their reasoning about this subset of agents, their reasoning about his reasoning about this subset, etc. This kind of unawareness was first discussed in Heifetz, Meier, and Schipper (2007) and is also found in Feinberg (2009).

Unawareness of higher-order reasoning prevents an agent from having beliefs beyond a certain order. It seems the least natural of the three kinds of unawareness considered here. The absence of beliefs beyond a certain order may be the result of other factors besides unawareness, e.g. computational limitations, time constraints or heuristic decision making. One instance where unawareness is a natural cause of bounded reasoning is when decision makers lack a theory of mind, such as animals or young children. In that case the agents are intrinsically unable to reason about the beliefs of others. The model of this kind of unawareness extends Heifetz and Kets (2011) to purely measurable spaces. The interpretation of Heifetz and Kets is bounded reasoning, not unawareness.

Modelling combined kinds of unawareness together with probabilistic beliefs is new in the literature. Many kinds of unawareness have been studied in propositional models using modal logic (Fagin and Halpern, 1988), but these models use possibility correspondences, not probabilities.

Another paper independently proving the existence of the universal type structure with propositional unawareness is Heifetz, Meier, and Schipper (2011c). Their definition of a type structure is the same (up to technical details) as in the present paper, because both extend Heifetz, Meier, and Schipper (2006) to probabilistic beliefs. A formal comparison is presented in Appendix B. The proof of universality in the present paper differs from Heifetz, Meier, and Schipper (2011c) in that no topology is required, but the universal type structure is obtained in a less explicit and constructive way.

[1]The author thanks an anonymous referee for this suggestion.

In games with unawareness, the question of robustness of predictions to uncertainty over the opponent's awareness of actions arises. Meier and Schipper (2012) use type structures with unawareness similar to the ones in the present paper to define Bayesian games with unawareness and examine the robustness question.

The universal type structure in this paper, like any model with unawareness, has to bypass the impossibility result of Dekel, Lipman, and Rustichini (1998) that standard state spaces preclude unawareness. Given natural properties of knowledge and unawareness, the result of Dekel, Lipman, and Rustichini rests on two basic axioms. The first axiom (real states) requires the negation and conjunction operators to behave in the standard way, and the second (event sufficiency) requires the operators for knowledge and awareness to take events to events. Knowledge and awareness of a statement cannot depend on other factors, such as the syntactic form of the statement. As pointed out by Heifetz, Meier, and Schipper (2006) and Schipper (2011), one of these axioms must be relaxed to model nontrivial unawareness. Economics models usually relax real states, while the logic literature mostly forgoes event sufficiency.

This paper follows the economics literature in omitting the real states axiom and keeping event sufficiency. This means negation and disjunction are nonstandard, but awareness depends only on the set of states belonging to an event, not on how the event is constructed or expressed.

The next section proves the existence of the universal type structure with propositional unawareness and shows how it may be unpacked into hierarchies of beliefs and awareness.

2. The existence of universal type structures with unawareness

In this section, propositional unawareness is added to the universal structure of belief hierarchies of Heifetz and Samet (1998b). Theirs is chosen as the basic framework, because unlike earlier models, it has no topological structure and only uses measurability assumptions. A topology of the type structure is not a necessary or natural component of descriptions of beliefs, as argued by Heifetz and Samet as well as Pintér (2010).

The existence proof of the universal type structure relies on the category-theoretic results of Moss and Viglizzo (2006); Viglizzo (2005), which use purely measurable spaces. If topological structure was added to make all spaces Polish, an alternate universality proof would use the results of Schubert (2008); Doberkat and Schubert (2011).

2.1. **Notation.** Before defining the universal type structure and proving its existence, some notation is needed. Fix for the rest of the paper a measurable space S as the space of primitive uncertainty and a finite set I as the set of agents (denoted $i, j \in I$). Nature is treated notationally as agent 0 and $I_0 = I \cup \{0\}$. Let $\bigsqcup$ denote disjoint union.

With propositional unawareness, the set of awareness levels $\mathbb{F}$ is a countable semilattice of σ-algebras on the space of primitive uncertainty. Elements of $\mathbb{F}$ are denoted by calligraphic letters $\mathcal{F}$, $\mathcal{E}$, $\mathcal{D}$. The semilattice operation is taking the coarsest common refinement. Since the semilattice is closed under coarsest common refinements of sets of σ-algebras, it is complete.[2] The notation $\mathcal{F} \triangleright \mathcal{E}$ means that $\mathcal{F}$ is finer than $\mathcal{E}$ and both belong to $\mathbb{F}$. The set S with the σ-algebra $\mathcal{F}$ is denoted $S_{\mathcal{F}}$.

For a measurable space M with σ-algebra $\mathcal{G}$, denote by $\Delta(M)$ the set of probability measures over X, endowed with the σ-algebra generated by

$$\{\beta^q(E) : E \in \mathcal{G},\ q \in [0,1]\}\,, \text{ where } \beta^q(E) = \{\mu \in \Delta(M) : \mu(E) \geq q\}\,.$$

As usual, δ_t denotes the Dirac delta function on $t \in M$, i.e. the probability distribution concentrated on t.

[2]The construction starting from a semilattice is not more general than starting from a lattice, since a complete semilattice is a lattice. The author thanks Aviad Heifetz for pointing this out.

The product of a vector of measurable spaces $(M_i)_{i\in I_0}$ is $\times_{i\in I_0} M_i = M$ and the product of all spaces except i in the vector is $\times_{j\in I_0\setminus\{i\}} M_i = M_{-i}$. A similar convention applies to collections of measurable spaces with more than one index $(M_{\mathcal{F},i})^{i\in I_0}_{\mathcal{F}\in\mathbb{F}}$, and to collections of measurable functions. The product over one index keeps the other index, $M_i = \times_{\mathcal{F}\in\mathbb{F}} M_{\mathcal{F},i}$, and the product over both indexes drops both, $M = \times_{\mathcal{F}\in\mathbb{F},\, i\in I_0} M_{\mathcal{F},i}$. Abusing notation, the collection $(M_{\mathcal{F},i})^{i\in I_0}_{\mathcal{F}\in\mathbb{F}}$ may also be denoted M. Products of measurable spaces have the product σ-algebra.

The σ-algebras on the space of primitive uncertainty are extended to the whole type structure by allowing agents with awareness $\mathcal{F}$ to distinguish only events in $\mathcal{F}$, others' beliefs differing on these events, their second order beliefs differing on beliefs differing on these events, etc.

2.2. **Definitions and existence.** The definitions of type structures with unawareness, type morphisms and the universal type structure with unawareness are presented next, followed by the existence proof. The type structure here is the belief analogue of the knowledge type structure of Heifetz, Meier, and Schipper (2006). Heifetz, Meier, and Schipper (2007) has the same definition as the present paper.[3]

The type structure in Definition 1 below models propositional unawareness when S is the set of states of nature (only contains payoff uncertainty). In that case, Definition 1 generalizes the Harsanyi type structure. If S consists of profiles of action sets of the players, the type structure describes unawareness of actions.

Definition 1. For the space of primitive uncertainty S, agents I and awareness levels $\mathbb{F}$, a type structure with propositional unawareness is $(M, g) = ((M_{\mathcal{F},i})^{i\in I_0}_{\mathcal{F}\in\mathbb{F}}, (g_{\mathcal{F},i})^{i\in I_0}_{\mathcal{F}\in\mathbb{F}})$ such that for each $\mathcal{F}\in\mathbb{F}$ and $i\in I$

(i) $M_{\mathcal{F},i}$ is a measurable space and $M_{\mathcal{F},0} = S_{\mathcal{F}}$,
(ii) $g_{\mathcal{F},0} : M_{\mathcal{F},0} \to S_{\mathcal{F}}$ is the identity,
(iii) $g_{\mathcal{F},i} : M_{\mathcal{F},i} \to \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta(M_{\mathcal{E}})$ consists of $g^{-i}_{\mathcal{F},i}$ and $g^{i}_{\mathcal{F},i}$, where
  (a) the function $g^{-i}_{\mathcal{F},i} : M_{\mathcal{F},i} \to \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta(M_{\mathcal{E},-i})$ is measurable,
  (b) for each $\mathcal{E}\lhd\mathcal{F}$ and $t\in M_{\mathcal{F},i}$ such that $g^{-i}_{\mathcal{F},i}(t)\in\Delta(M_{\mathcal{E},-i})$, the function $g^{i}_{\mathcal{F},i} : M_{\mathcal{F},i} \to \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta(M_{\mathcal{E},i})$ maps t to $\delta_{t_{\mathcal{E},i}} \in \Delta(M_{\mathcal{E},i})$ with $g^{-i}_{\mathcal{E},i}(t_{\mathcal{E},i}) = g^{-i}_{\mathcal{F},i}(t)$.

Each element $M_{\mathcal{F},i}$ of the type structure collects the types of agent i whose awareness level is $\mathcal{F}$ or coarser. The component $g_{\mathcal{F},i}$ of i's type function maps each type in $M_{\mathcal{F},i}$ to its belief, which is over $M_{\mathcal{E}}$ for some $\mathcal{E}\lhd\mathcal{F}$. The element $M_{\mathcal{E}}$ of the type structure on which i's belief is defined is one-to-one with i's awareness level $\mathcal{E}$. The final part of the definition ensures that each agent is certain of his own belief.

The type function g_i for agent i is $(g_{\mathcal{F},i})_{\mathcal{F}\in\mathbb{F}}$. For ease of notation, the domain of g_i may be written as M, with the understanding that g_i only depends on the M_i coordinate.

In order to familiarize the reader with the definition of type structures, the structure corresponding to the game of Feinberg (2009) in Fig. 1 is subsequently discussed and illustrated in Fig. 2. Since the game in question only features uncertainty about available actions, the elements of the space of primitive uncertainty are profiles of players' action sets. Empty action sets are not allowed. The two players R and C each have three actions, so $2^3 - 1 = 7$ conceivable action sets. This means 49 profiles of action sets are possible. The ones relevant to the game are $s_1 = \{a_1, a_2, a_3\} \times \{b_1, b_2, b_3\}$ and $s_2 = \{a_1, a_2\} \times \{b_1, b_2, b_3\}$. The space of primitive uncertainty is $S = \{s_1, s_2\}$.

There are two awareness levels, $\mathcal{F} = \{\emptyset, \{s_1\}, \{s_2\}, \{s_1, s_2\}\}$ and $\mathcal{E} = \{\emptyset, \{s_1, s_2\}\}$. An agent with awareness level $\mathcal{E}$ is unaware of the distinction between s_1 and s_2. This can be interpreted as unawareness of the existence of a_3. The type set for player R at awareness level $\mathcal{F}$ is $M_{\mathcal{F},R} =$

[3]The author thanks Aviad Heifetz for pointing this out.

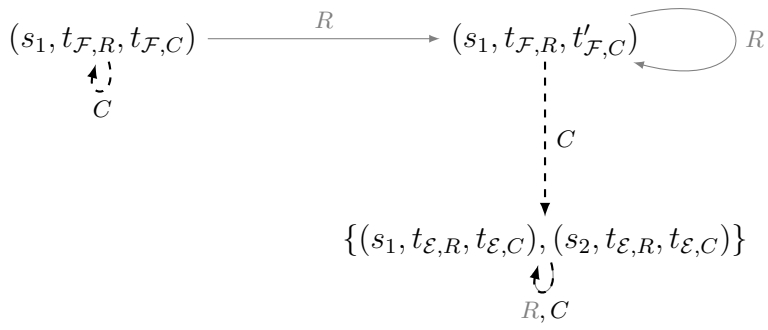

FIGURE 2. A type structure with unawareness corresponding to Feinberg's game.

$\{t_{\mathcal{F},R}\}$ and at level $\mathcal{E}$, it is $M_{\mathcal{E},R} = \{t_{\mathcal{E},R}\}$. The type set for player C at awareness level $\mathcal{F}$ is $M_{\mathcal{F},C} = \{t_{\mathcal{F},C}, t'_{\mathcal{F},C}\}$ and at level $\mathcal{E}$, it is $M_{\mathcal{E},C} = \{t_{\mathcal{E},C}\}$. The game in Fig. 1 can be represented with four full types: $(s_1, t_{\mathcal{F},R}, t_{\mathcal{F},C})$ and $(s_1, t_{\mathcal{F},R}, t'_{\mathcal{F},C})$ in $M_{\mathcal{F}}$ and $(s_1, t_{\mathcal{E},R}, t_{\mathcal{E},C})$ and $(s_2, t_{\mathcal{E},R}, t_{\mathcal{E},C})$ in $M_{\mathcal{E}}$.

The type function g_R for player R satisfies

$$g_R(t_{\mathcal{F},R})(\{(s_1, t_{\mathcal{F},R}, t'_{\mathcal{F},C})\}) = 1 \text{ and } g_R(t_{\mathcal{E},R})(\{(s_1, t_{\mathcal{E},R}, t_{\mathcal{E},C}), (s_2, t_{\mathcal{E},R}, t_{\mathcal{E},C})\}) = 1. \tag{1}$$

The type function g_C for C satisfies

$$\begin{aligned} &g_C(t_{\mathcal{F},C})(\{(s_1, t_{\mathcal{F},R}, t_{\mathcal{F},C})\}) = 1, \\ &g_C(t'_{\mathcal{F},C})(\{(s_1, t_{\mathcal{E},R}, t_{\mathcal{E},C}), (s_2, t_{\mathcal{E},R}, t_{\mathcal{E},C})\}) = 1 \text{ and} \\ &g_C(t_{\mathcal{E},C})(\{(s_1, t_{\mathcal{E},R}, t_{\mathcal{E},C}), (s_2, t_{\mathcal{E},R}, t_{\mathcal{E},C})\}) = 1. \end{aligned} \tag{2}$$

In Fig. 2, the arrows correspond to the above conditions on the type functions, e.g. the horizontal arrow labelled R means that at $(s_1, t_{\mathcal{F},R}, t_{\mathcal{F},C})$, player R's type function g_R puts probability one on $\{(s_1, t_{\mathcal{F},R}, t'_{\mathcal{F},C})\}$. The interpretation of the other arrows is similar.

Condition (iii)(b) of Definition 1 (certainty in one's own belief) is reflected in Fig. 2 by the property that at any state of the world t, each player only puts positive probability on states of the world t' where his or her own belief is the same as at t. For example, at $(s_1, t_{\mathcal{F},R}, t_{\mathcal{F},C})$, player R has type $t_{\mathcal{F},R}$ and puts positive probability only on $(s_1, t_{\mathcal{F},R}, t'_{\mathcal{F},C})$, where R's type is also $t_{\mathcal{F},R}$. Diagrammatically, at any type in an event to which an arrow of player i points, i's arrow makes a loop back to that event.

In the above example, primitive uncertainty is only about the availability of actions. Payoff uncertainty could be added by specifying an appropriate internal structure for S. For each agent i, denote the set of payoff types by Θ_i and the set of possible action sets by Σ_i, then $S = \times_{i \in I} (\Theta_i \times \Sigma_i)$ describes all the utility functions and action sets of the agents.

To connect type structures with unawareness to each other, type morphisms are needed. These are maps from one type structure to another that preserve belief and awareness information, as detailed in Definition 2.

Definition 2. A type morphism between (M', g') and (M, g) is a vector $f = (f_{\mathcal{F},i})_{\mathcal{F} \in \mathbb{F}}^{i \in I_0}$ of measurable functions $f_{\mathcal{F},i} : M'_{\mathcal{F},i} \to M_{\mathcal{F},i}$, such that

$$g_i(f(t'))(E) = g'_i(t')(f_{\mathcal{D}}^{-1}(E)) \quad \forall i \in I \quad \forall t' \in M' \quad \forall \mathcal{D} \in \mathbb{F} \quad \forall E \subseteq M_{\mathcal{D}} \text{ measurable.} \tag{3}$$

A type morphism f is an isomorphism if it is a measure-preserving bijection between M' and M, i.e. $f^{-1} : M \to M'$ is also a type morphism.

The definition of a universal type structure is standard—it is a type structure containing all other type structures. Since propositional unawareness is assumed in this section, the set of awareness levels $\mathbb{F}$ is a countable lattice of σ-algebras on the space of primitive uncertainty.

Definition 3. A type structure with unawareness (Ω, g) is universal if for every type structure (M', g') with the same set of agents I, the same set of awareness levels $\mathbb{F}$ and the same space of primitive uncertainty S there is a unique type morphism from (M', g') to (Ω, g).

The universal type structure in the present section is the set of all hierarchies of belief and awareness that some type in some type structure maps to. The set of all hierarchies is constructed in Subsection 2.3. For propositional unawareness, Heifetz, Meier, and Schipper (2011c) independently prove the existence of a universal type structure by a different method—applying a generalized Kolmogorov Extension Theorem. For this they need the assumptions that the space of primitive uncertainty is a Hausdorff topological space and the (Borel) probability measures are compact regular.

The existence and uniqueness of a universal type structure for propositional unawareness is proved in Theorem 1.

Theorem 1. *Fixing the space of primitive uncertainty S, the set of agents I and the set of awareness levels $\mathbb{F}$, there exists a universal type structure with propositional unawareness (Ω, g) unique up to isomorphism. In (Ω, g), the type function g is an isomorphism.*

The proof in the appendix translates the problem into category theory and uses Viglizzo (2005), which generalizes Heifetz and Samet (1998b). The theorem in this section differs from Heifetz, Meier, and Schipper (2011c) because it does not use any topological assumptions. A similar proof technique is used in Pintér and Udvari (2011). For the case without unawareness, the fact that the type function is an isomorphism is proved in Meier (2012) by a different method than the one used in the present paper. The type function being an isomorphism implies the type structure is belief-complete[4], defined as all type functions being onto. An interpretation is that for every combination of belief and awareness there exists a type with that combination. The isomorphism result is a stronger statement: for every combination of belief and awareness there is a unique type with that combination.

2.3. **From types to belief hierarchies with unawareness.** Two proofs of the existence of the universal structure are given in Viglizzo (2005)—one constructs the structure so that a type equals the set of all events which contain the type, the other shows that a subset of the set of all hierarchies is the universal type structure. The present subsection unpacks types (of any type structure with the appropriate set of agents and awareness levels and the appropriate space of primitive uncertainty) into hierarchies of belief and awareness.

First the set of all hierarchies is defined and then the function from types to hierarchies is given. The construction is similar to Heifetz and Samet (1998b). The base case of the inductive construction of hierarchies of belief and awareness is setting $H^0_{\mathcal{F},i}$ to be a singleton for all $i \in I$ and $\mathcal{F} \in \mathbb{F}$, and setting $H^k_{\mathcal{F},0} = S_{\mathcal{F}}$ for all $k \geq 0$ and all $\mathcal{F} \in \mathbb{F}$.

The inductive step consists of setting $H^{k+1}_{\mathcal{F},0} = S_{\mathcal{F}}$ and for $i \neq 0$,

$$H^{k+1}_{\mathcal{F},i} = \bigsqcup_{\mathcal{E} \lhd \mathcal{F}} \Delta\left(H^k_{\mathcal{E}}\right).$$

Agent i at awareness $\mathcal{F}$ has the set of hierarchies $H_{\mathcal{F},i} = \times_{k \geq 0} H^k_{\mathcal{F},i}$. For each k there is a natural projection from $H_{\mathcal{F},i}$ to $H^k_{\mathcal{F},i}$. The set of hierarchies of all players at all awareness levels is $H =$

[4]The concept of belief-completeness was introduced by Brandenburger (2003) for possibility structures. The author thanks an anonymous referee for pointing this out.

$\left(S_{\mathcal{F}}, (H_{\mathcal{F},i})^{i\in I}\right)_{\mathcal{F}\in\mathbb{F}}$, with natural projections to the set of order-k hierarchies for all k. The universal type structure Ω is mapped to a strict subset of H. The construction of the mapping ensures that all elements in the image of any type structure in H are coherent, i.e. beliefs of different orders do not contradict each other.

Given any type structure (M, g) satisfying Definition 1, the mapping $h = (h_{\mathcal{F},i})^{i\in I_0}_{\mathcal{F}\in\mathbb{F}} : M \to H$ from types to hierarchies of beliefs and awareness is constructed inductively. To define $h_{\mathcal{F},i}$, first $h^k_{\mathcal{F},i} : M_{\mathcal{F}} \to H^k_{\mathcal{F},i}$ is defined for every k. Let $h^k_{\mathcal{F},0} = g_{\mathcal{F},0}$ for every k. For $i \in I$, $h^0_{\mathcal{F},i}$ is uniquely defined as the constant function mapping to the singleton $H^0_{\mathcal{F},i}$. Denote $(h^k_{\mathcal{E},j})^{j\in I_0}$ by $h^k_{\mathcal{E}}$ and inductively define $h^{k+1}_{\mathcal{F},i}$ by the condition that if $g_i(t) \in \Delta(M_{\mathcal{E}})$, then $h^{k+1}_{\mathcal{F},i}(t) \in \Delta(H^k_{\mathcal{E}})$ and

$$h^{k+1}_{\mathcal{F},i}(t)(F) = g_i(t)((h^k_{\mathcal{E}})^{-1}(F)) \quad \forall t \in M \quad \forall F \subseteq H^k_{\mathcal{E}} \text{ measurable}, \tag{4}$$

where $(h^k_{\mathcal{E}})^{-1}(F)$ gives the types in $M_{\mathcal{E}}$ (but not in $M_{\mathcal{E}'}$ for $\mathcal{E}' \neq \mathcal{E}$) that map to hierarchies in F. For each $\mathcal{F} \in \mathbb{F}$ and $i \in I$, define $h_{\mathcal{F},i} = (h^k_{\mathcal{F},i})_{k\geq 0}$ and $h_{\mathcal{F},0} = g_{\mathcal{F},0}$. This completes the construction of h.

The hierarchy description map is illustrated using the type structure in Fig. 2. Abbreviate the types as $t_{\mathcal{F}} = (s_1, t_{\mathcal{F},R}, t_{\mathcal{F},C})$, $t'_{\mathcal{F}} = (s_1, t_{\mathcal{F},R}, t'_{\mathcal{F},C})$, $t^1_{\mathcal{E}} = (s_1, t_{\mathcal{E},R}, t_{\mathcal{E},C})$ and $t^2_{\mathcal{E}} = (s_2, t_{\mathcal{E},R}, t_{\mathcal{E},C})$. Let $H^0_{\mathcal{F},R} = H^0_{\mathcal{F},C} = \{\alpha_{\mathcal{F}}\}$ and $H^0_{\mathcal{E},R} = H^0_{\mathcal{E},C} = \{\alpha_{\mathcal{E}}\}$, where $\alpha_{\mathcal{F}}$, $\alpha_{\mathcal{E}}$ are just notation to express that the zeroth order hierarchy sets for the players are singletons. For all k, $H^k_{\mathcal{F},0} = S_{\mathcal{F}}$, which is S with the discrete σ-algebra, and $H^k_{\mathcal{E},0} = S_{\mathcal{E}}$, which has the σ-algebra $\{\emptyset, \{s_1, s_2\}\}$. Then $H^0_{\mathcal{F}} = S_{\mathcal{F}} \times \{\alpha_{\mathcal{F}}\} \times \{\alpha_{\mathcal{F}}\}$ and $H^0_{\mathcal{E}} = S_{\mathcal{E}} \times \{\alpha_{\mathcal{E}}\} \times \{\alpha_{\mathcal{E}}\}$.

For all k, $h^k_{\mathcal{F},0}(t_{\mathcal{F}}) = h^k_{\mathcal{F},0}(t'_{\mathcal{F}}) = h^k_{\mathcal{E},0}(t^1_{\mathcal{E}}) = s_1$ and $h^k_{\mathcal{E},0}(t^2_{\mathcal{F}}) = s_2$. The zeroth order hierarchy maps for the players are $h^0_{\mathcal{F},R}(t_{\mathcal{F}}) = h^0_{\mathcal{F},R}(t'_{\mathcal{F}}) = h^0_{\mathcal{F},C}(t_{\mathcal{F}}) = h^0_{\mathcal{F},C}(t'_{\mathcal{F}}) = \alpha_{\mathcal{F}}$ and $h^0_{\mathcal{E},R}(t^1_{\mathcal{E}}) = h^0_{\mathcal{E},R}(t^2_{\mathcal{E}}) = h^0_{\mathcal{E},C}(t^1_{\mathcal{E}}) = h^0_{\mathcal{E},C}(t^2_{\mathcal{E}}) = \alpha_{\mathcal{E}}$. Overall, $h^0_{\mathcal{F}}(t_{\mathcal{F}}) = (s_1, \alpha_{\mathcal{F}}, \alpha_{\mathcal{F}}) = h^0_{\mathcal{F}}(t'_{\mathcal{F}})$, $h^0_{\mathcal{E}}(t^1_{\mathcal{E}}) = (s_1, \alpha_{\mathcal{E}}, \alpha_{\mathcal{E}})$ and $h^0_{\mathcal{E}}(t^2_{\mathcal{E}}) = (s_2, \alpha_{\mathcal{E}}, \alpha_{\mathcal{E}})$.

As to the first order beliefs, consider the event $F = \{(s_1, \alpha_{\mathcal{F}}, \alpha_{\mathcal{F}})\} \subseteq H^0_{\mathcal{F}}$. Based on the zeroth order beliefs, $(h^0_{\mathcal{F}})^{-1}(F) = \{t_{\mathcal{F}}, t'_{\mathcal{F}}\}$, so using (4), $h^1_{\mathcal{F},R}(t_{\mathcal{F}})(F) = g_R(t_{\mathcal{F}})(\{t_{\mathcal{F}}, t'_{\mathcal{F}}\})$, which by (1) equals one. It turns out that $h^1_{\mathcal{F},R}(t'_{\mathcal{F}}) = h^1_{\mathcal{F},C}(t_{\mathcal{F}}) = h^1_{\mathcal{F},R}(t_{\mathcal{F}})$, reflecting the fact that these players at these types have the same belief about primitive uncertainty. However, $h^1_{\mathcal{F},C}(t'_{\mathcal{F}})$ has a lower awareness level, so $h^1_{\mathcal{F},C}(t'_{\mathcal{F}})(H^0_{\mathcal{E}}) = g_C(t'_{\mathcal{F}})(\{t^1_{\mathcal{E}}, t^2_{\mathcal{E}}\}) = 1$, based on (2) and the fact that $(h^0_{\mathcal{E}})^{-1}(H^0_{\mathcal{E}}) = \{t^1_{\mathcal{E}}, t^2_{\mathcal{E}}\}$. The beliefs $h^1_{\mathcal{E},i}(t^n_{\mathcal{E}})$ for $i = R, C$ and $n = 1, 2$ are the same as $h^1_{\mathcal{F},C}(t'_{\mathcal{F}})$.

For the second order beliefs, look at the events $E = \left\{(s_1, h^1_{\mathcal{F},R}(t_{\mathcal{F}}), h^1_{\mathcal{F},C}(t_{\mathcal{F}}))\right\} \subseteq H^1_{\mathcal{F}}$ and $E' = \left\{(s_1, h^1_{\mathcal{F},R}(t'_{\mathcal{F}}), h^1_{\mathcal{F},C}(t'_{\mathcal{F}}))\right\} \subseteq H^1_{\mathcal{F}}$. On these, $h^2_{\mathcal{F},R}(t_{\mathcal{F}})(E') = g_R(t_{\mathcal{F}})((h^1_{\mathcal{F}})^{-1}(E')) = g_R(t_{\mathcal{F}})(\{t'_{\mathcal{F}}\}) = 1$ and $h^2_{\mathcal{F},C}(t_{\mathcal{F}})(E) = g_C(t_{\mathcal{F}})((h^1_{\mathcal{F}})^{-1}(E)) = g_C(t_{\mathcal{F}})(\{t_{\mathcal{F}}\}) = 1$ by (1) and (2).

The third order beliefs $h^3_{\mathcal{F},C}(t_{\mathcal{F}}) \in \Delta(H^2_{\mathcal{F}})$ put probability one on $\left\{(s_1, h^2_{\mathcal{F},R}(t_{\mathcal{F}}), h^2_{\mathcal{F},C}(t_{\mathcal{F}}))\right\} \subseteq H^2_{\mathcal{F}}$, in which $h^2_{\mathcal{F},R}(t_{\mathcal{F}})$ is certain of E', while at E', $h^1_{\mathcal{F},C}(t'_{\mathcal{F}})$ puts probability 1 on $H^0_{\mathcal{E}}$. This belief describes how in Fig. 1 at type $t_{\mathcal{F}}$ (at the top left normal form), C believes that R believes the correct type is $t'_{\mathcal{F}}$ (corresponding to the top right normal form), while at $t'_{\mathcal{F}}$, C is unaware of action a_3 and believes the real game is the bottom right normal form, at which the beliefs of the players are summarized by $t_{\mathcal{E},R}$, $t_{\mathcal{E},C}$.

Continuing the above process, further orders of beliefs can be described, but these are omitted from the example to save space.

The hierarchy description map h associates with every type t of every type structure (M, g) a hierarchy of beliefs and awareness. Slightly extending the argument in Heifetz and Samet (1998b), Section 5, or using Viglizzo (2005), Chapter 5, it can be shown that one way to define the universal structure is as those elements of H to which some type in some type structure with unawareness is mapped by the h defined on that type structure. The argument proceeds as follows. The first step

is to prove that any type morphism preserves the hierarchical description of a type: if $f : M \to M'$ is a type morphism and h, h' are the hierarchy description maps for M, M' respectively, then $h'_{\mathcal{F},i}(f(t)) = h_{\mathcal{F},i}(t)$ for all $\mathcal{F} \in \mathbb{F}$, $i \in I_0$ and $t \in M$.

Next, a structure (Ω^*, g^*) is defined. Ω^* is the set of all hierarchies $t^*_{\mathcal{F},i} \in H_{\mathcal{F},i}$ for which there exists some type t in some type structure (M, g) such that $h_{\mathcal{F},i}(t) = t^*_{\mathcal{F},i}$. The type function $g^*_{\mathcal{F},i}$ is defined by referring back to the type structure and type from which the hierarchy was generated, as in (4): $g^*_{\mathcal{F},i}(t^*_{\mathcal{F},i})(F) = g_{\mathcal{F},i}(t)(h_{\mathcal{E}})^{-1}(F))$ for $\mathcal{E} \lhd \mathcal{F}$, $F \subseteq H_{\mathcal{E}}$. Proving that (Ω^*, g^*) is a type structure is a matter of showing that it inherits the properties of the type structures used in defining it.

The argument continues by demonstrating that for every type structure (M, g), the hierarchy description map is a type morphism from (M, g) to (Ω^*, g^*). The proof is by verifying the commutativity condition in the definition of a type morphism.

The final step is showing that the map $h : \Omega^* \to \Omega^*$ is the identity. This is proved by checking that for every i, $\mathcal{F}$ and k, $h^k_{\mathcal{F},i}$ is the projection from $\Omega^*_{\mathcal{F},i}$ to $H^k_{\mathcal{F},i}$. The result that h is the identity on Ω^* implies that (Ω^*, g^*) is universal—every type structure maps into it via a unique type morphism.

In the topological framework used by Battigalli and Siniscalchi (1999) and Heifetz, Meier, and Schipper (2011c), the proof of universality is similar to the preceding argument. First the type set is constructed. Next the hierarchy description map from any type structure to the (candidate) universal one is shown to be a type morphism. The uniqueness of the type morphism is then proved, which establishes universality.

The main difference in the topological case is that the type set of the universal structure is constructed explicitly from coherent hierarchies of beliefs (and awareness, in the case of Heifetz, Meier, and Schipper (2011c)). The type function is then a homeomorphism. In the purely measurable case, the type function of the universal structure is built from the hierarchy description map and the type functions of arbitrary type structures. A comparison to the topological approach of Heifetz, Meier, and Schipper (2011b) is presented in the appendix.

3. Other kinds of unawareness

Unawareness in the universal type structure does not have to be propositionally generated, as in the preceding section. The same proof of existence holds when the awareness levels are subsets J of the finite set I of agents. In that case, an agent could reason only about the agents he is aware of, their reasoning about the agents he is aware of etc.

Bounded reasoning in the style of Heifetz and Kets (2011) can be expressed as unawareness of higher-order reasoning by taking the awareness levels to be $1 \leq k \leq \infty$. An agent with awareness k can reason about order $k-1$ and lower order beliefs of other agents, their reasoning about order $k-2$ and lower orders etc. The special case of infinite depth of reasoning is taken care of by assuming $\infty - 1 = \infty$. In the type structure for bounded reasoning below, positive introspection is allowed, so an agent has all orders of beliefs about himself, but only finite orders about other agents.

The proof of existence of the universal structure in Viglizzo (2005) and Moss and Viglizzo (2006) only deals with a finite number of agents, and mathematically the fixed set of awareness levels in this paper behaves like the set of agents. With unawareness of higher-order reasoning, the set of awareness levels is countably infinite. However, since all the proofs of Moss and Viglizzo (2006) use measurable spaces and the assumption about a finite number of agents is only used to make showing measurability easier, the proofs work with a countable number of awareness levels as well.[5]

[5]This was confirmed by Larry Moss in a private communication.

If the existence result holds for several kinds of unawareness, it also holds for their combination, with only notational changes in the proof. Type structures with three kinds of unawareness consist of measurable spaces $M_{\mathcal{F},J,k,i}$ one for each agent i and awareness level $(\mathcal{F}, J, k)$. The triple consists of a σ-algebra $\mathcal{F} \in \mathbb{F}$ on the space of primitive uncertainty, a set of agents $J \subseteq I$ and an order of reasoning k. Each type in $M_{\mathcal{F},J,k,i}$ can reason only about the agents in J and events in $\mathcal{F}$, and the reasoning can extend only up to order k.

The definitions of type structures with different kinds of unawareness are given next, followed by the existence theorem of the universal type structure for all of them. The definition of the universal type structure is the same for all kinds of unawareness—it is the structure that has a unique type morphism from every type structure with the same kind of unawareness into it. The definition of a type morphism must be tailored to each type structure in the three subsequent definitions. The essence remains the same as Definition 2, but the indexes of the components of the vector of functions change.

In the following definitions, $\mathcal{G}$ is the finest σ-algebra in $\mathbb{F}$.

Definition 4. A type structure with unawareness of agents is $(M, g) = \left(M_0, (M_{J,i})_{J\subseteq I}^{i\in J},\ g_0, (g_{J,i})_{J\subseteq I}^{i\in J}\right)$ such that for each J and i,

(i) $M_{J,i}$ is a measurable space and $M_0 = S_{\mathcal{G}}$,
(ii) $g_0 : M_0 \to S_{\mathcal{G}}$ is the identity,
(iii) the type function $g_{J,i} : M_{J,i} \to \bigsqcup_{i\in J'\subseteq J} \Delta\left(M_0 \times \times_{j\in J'} M_{J',j}\right)$ consists of $g_{J,i}^{-i}$ and $g_{J,i}^{i}$, where
  (a) the function $g_{J,i}^{-i} : M_{J,i} \to \bigsqcup_{i\in J'\subseteq J} \Delta\left(M_0 \times \times_{j\in J'\setminus\{i\}} M_{J',j}\right)$ is measurable and
  (b) for each $J' \subseteq J$ and $t \in M_{J,i}$ such that $g_{J,i}^{-i}(t) \in \Delta(M_{J',-i})$, the function $g_{J,i}^{i} : M_{J,i} \to \bigsqcup_{i\in J'\subseteq J} \Delta(M_{J',i})$ maps t to $\delta_{t_{J',i}} \in \Delta(M_{J',i})$ with $g_{J',i}^{-i}(t_{J',i}) = g_{J,i}^{-i}(t)$.

Definition 5. A type structure with unawareness of higher-order reasoning is

$$(M, g) = \left(M_0, (M_{k,i})_{1\le k\le\infty}^{i\in I},\ g_0, (g_{k,i})_{1\le k\le\infty}^{i\in I}\right)$$

such that for each $1 \le k \le \infty$ and i,

(i) $M_{k,i}$ is a measurable space, $M_{0,i}$ is a singleton, and $M_0 = S_{\mathcal{G}}$,
(ii) $g_0 : M_0 \to S_{\mathcal{G}}$ is the identity,
(iii) the type function $g_{k,i} : M_{k,i} \to \Delta M_{k,i} \times \bigsqcup_{n\le k-1} \Delta\left(M_0 \times \times_{j\neq i} M_{n,j}\right)$ consists of $g_{k,i}^{-i}$ and $g_{k,i}^{i}$, where
  (a) $g_{k,i}^{-i} : M_{k,i} \to \bigsqcup_{n\le k-1} \Delta\left(M_0 \times \times_{j\neq i} M_{n,j}\right)$ is measurable and
  (b) $g_{k,i}^{i}(t) = \delta_t$.

The condition $g_{k,i}^{i}(t) = \delta_t$ in the preceding definition ensures positive introspection—the agent is always certain of his own type. With bounded reasoning, positive introspection may not be a natural condition, but a lack of positive introspection may also cause conceptual problems. Other definitions of a type structure with bounded reasoning are possible, e.g. that of Heifetz and Kets (2011). The ideas of the present paper can be used to analyze a wide variety of type structures. The problem with positive introspection in type structures with bounded reasoning is about interpretation, not about mathematics.

Definition 6. A type structure with three kinds of unawareness is

$$(M, g) = \left(\left(M_{\mathcal{F},0}, (M_{\mathcal{F},J,k,i})_{1\le k\le\infty}^{i\in J\subseteq I}\right)_{\mathcal{F}\in\mathbb{F}}, \left(g_{\mathcal{F},0}, (g_{\mathcal{F},J,k,i})_{1\le k\le\infty}^{i\in J\subseteq I}\right)_{\mathcal{F}\in\mathbb{F}}\right)$$

such that for each $\mathcal{F}$, J, k and i,

(i) $M_{\mathcal{F},J,k,i}$ is a measurable space, $M_{\mathcal{E},J',0,i}$ is a singleton, and $M_{\mathcal{F},0} = S_{\mathcal{F}}$,
(ii) $g_{\mathcal{F},0} : M_{\mathcal{F},0} \to S_{\mathcal{F}}$ is the identity,

(iii) the type function

$$g_{\mathcal{F},J,k,i} : M_{\mathcal{F},J,k,i} \to \bigsqcup_{\substack{\mathcal{E} \lhd \mathcal{F} \\ i \in J' \subseteq J}} \left(\Delta M_{\mathcal{E},J',k,i} \times \bigsqcup_{n \leq k-1} \Delta \left(M_{\mathcal{E},0} \times \times_{j \in J' \setminus \{i\}} M_{\mathcal{E},J',n,j} \right) \right)$$

consists of $g^{-i}_{\mathcal{F},J,k,i}$ and $g^{i}_{\mathcal{F},J,k,i}$, where
(a)

$$g^{-i}_{\mathcal{F},J,k,i} : M_{\mathcal{F},J,k,i} \to \bigsqcup_{\substack{\mathcal{E} \lhd \mathcal{F} \\ i \in J' \subseteq J \\ n \leq k-1}} \Delta \left(M_{\mathcal{E},0} \times \times_{j \in J' \setminus \{i\}} M_{\mathcal{E},J',n,j} \right) \text{ is measurable and}$$

(b) $g^{i}_{\mathcal{F},J,k,i}(t) = \delta_{t_{\mathcal{E},J',k,i}} \in \Delta(M_{\mathcal{E},J',k,i})$, where $t_{\mathcal{E},J',k,i}$ is such that $g^{-i}_{\mathcal{E},J',k,i}(t_{\mathcal{E},J',k,i}) = g^{-i}_{\mathcal{F},J,k,i}(t)$.

Theorem 2. *Fixing the space of primitive uncertainty, the set of agents and the set of awareness levels, there exists a universal type structure with unawareness of agents* (Ω^a, g^a)*, a universal type structure with unawareness of higher-order reasoning* (Ω^b, g^b) *and a universal type structure with three kinds of unawareness* (Ω^c, g^c)*. All universal type structures are unique up to isomorphism. The type functions in the universal type structures are isomorphisms.*

The proof in the appendix is very similar to Theorem 1. It also translates the problem to category theory and uses the results of Viglizzo (2005).

4. Conclusion

This paper proves the existence of the universal type structure with three kinds of unawareness and their combination in the purely measurable case. Unawareness type structures have been used to prove no-trade and agreement theorems (Heifetz, Meier, and Schipper, 2007) and study the robustness of predictions in Bayesian games with unawareness to uncertainty over the opponent's awareness of actions (Meier and Schipper, 2012). Other potential applications of universal type structures with unawareness mirror those of universal belief type structures—studying common priors (Mertens and Zamir, 1985), common certainty of the structure (Brandenburger and Dekel, 1993) and robustness of solution concepts to perturbations of higher-order beliefs (Weinstein and Yildiz, 2007; Ely and Peski, 2011; Chen, Di Tillio, Faingold, and Xiong, 2010). Type structures with unawareness enable games with unawareness to be analyzed, in addition to standard games. The type structures in the present paper only apply to static games. To study extensive form games with unawareness, type structures with conditional probability and unawareness would have to be constructed.

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Appendix A. Proofs

To prove the existence of a universal type structure with unawareness using Viglizzo (2005), the problem is restated in category-theoretic terms, as in Moss (2011). The following concepts from category theory are needed.

A category is a directed graph consisting of objects (nodes) and morphisms (arrows) in which composition is defined and there are identity edges. In this paper, only the category $\texttt{Meas}^{N\times I_0}$, consisting of vectors of measurable spaces as objects and vectors of measurable functions as morphisms, is needed. I_0 is the set of agents together with nature and N is the cardinality of the set of awareness levels. A functor $V : \texttt{Meas}^{N\times I_0} \to \texttt{Meas}^{N\times I_0}$ maps each object $M \in \texttt{Meas}^{N\times I_0}$ to an object $V(M) \in \texttt{Meas}^{N\times I_0}$ and each morphism $f : M' \to M$ to a morphism $V(f) : V(M') \to V(M)$, preserving compositions of morphisms, so $V(f \circ g) = V(f) \circ V(g)$.

A coalgebra (M, g) for a functor $V : \texttt{Meas}^{N\times I_0} \to \texttt{Meas}^{N\times I_0}$ is a vector of measurable spaces M and a vector of measurable functions $g : M \to V(M)$. A coalgebra morphism from (M', g') to (M, g) is a morphism $f : M' \to M$ that satisfies $g \circ f = V(f) \circ g'$, as illustrated in the following diagram.

$$\begin{tikzcd} M' \arrow[r, "g'"] \arrow[d, "f"] & V(M') \arrow[d, "V(f)"] \\ M \arrow[r, "g"] & V(M) \end{tikzcd}$$

A final coalgebra for V is one that has a unique coalgebra morphism from all coalgebras of V into it. The type structures of Heifetz and Samet (1998b) are coalgebras for the functor $W : \texttt{Meas}^I \to \texttt{Meas}^I$, defined as $W(M) = (\Delta(S \times M_{-i}))_{i\in I}$, and the universal type structure is a final coalgebra of W (Moss, 2011).

The next two lemmas characterize the type structures with propositional unawareness as coalgebras of a certain functors in $\texttt{Meas}^{N\times I_0}$.

Lemma 3. *A type structure with propositional unawareness and awareness levels $\mathbb{F}$ is a coalgebra for the functor $V : \texttt{Meas}^{N\times I_0} \to \texttt{Meas}^{N\times I_0}$ defined as $V = (V_{\mathcal{F},i})^{i\in I_0}_{\mathcal{F}\in\mathbb{F}}$, where for $i \in I$, $V_{\mathcal{F},i}(M) = \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} (\Delta(M_{\mathcal{E},-i}) \times \Delta(M_{\mathcal{E},i}))$ and $V_{\mathcal{F},0}(M) = S_{\mathcal{F}}$.*

Proof. By Definition 1, a type structure with awareness is an (object, morphism) pair (M, g) in $\texttt{Meas}^{N\times I_0}$, where the morphism g maps M to $V(M) = \left(S_{\mathcal{F}}, (\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} (\Delta(M_{\mathcal{E},-i}) \times \Delta(M_{\mathcal{E},i})))^{i\in I}\right)_{\mathcal{F}\in\mathbb{F}}$ in $\texttt{Meas}^{N\times I_0}$. So g maps M to its image under the functor V. Therefore (M, g) is a coalgebra for the functor. □

Lemma 4. *Coalgebra morphisms are type morphisms. Final coalgebras are universal type structures.*

Proof. Coalgebra morphisms in $\texttt{Meas}^{N\times I_0}$ and type morphisms are vectors of measurable functions. A coalgebra morphism $f : M' \to M$ for the functor V satisfies $g \circ f = V(f) \circ g'$, meaning that for each $t' \in M'$, the same element of $V(M)$ results from applying $g \circ f$ as from $V(f) \circ g'$. This is exactly Eq. (3) in Definition 2.

By Lemma 3, type structures are coalgebras for $V(M) = \left(S_{\mathcal{F}}, (\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} (\Delta(M_{\mathcal{E},-i}) \times \Delta(M_{\mathcal{E},i})))^{i\in I}\right)_{\mathcal{F}\in\mathbb{F}}$. The definition of a universal type structure is that there exists a unique type morphism from any type structure into it, and the definition of a final coalgebra is that there exists a unique coalgebra morphism from any coalgebra into it. Since type morphisms are coalgebra morphisms, universal type structures are final coalgebras. □

The result that the type function in the universal type structure is a measure-theoretic isomorphism relies on Lambek's Lemma (Lambek, 1968, Lemma 3.5), an important result in category theory. The statement of Lambek's Lemma adapted to the present context as Theorem 2.3 in Viglizzo (2005) is given next, followed by the proof of Theorem 1.

Lemma 5 (Lambek's Lemma, Theorem 2.3 in Viglizzo (2005))**.** *If (Ω, g) is a final coalgebra of the functor V, then g is an isomorphism between Ω and $V(\Omega)$.*

Proof of Theorem 1. According to Viglizzo (2005, Theorem 7.1), any functor on $\texttt{Meas}^I$ composed of Δ, Cartesian products, disjoint unions, the identity, constant functors and projections from $\texttt{Meas}^I$ to $\texttt{Meas}$ has a final coalgebra. Viglizzo's proof remains unchanged if the functor is on $\texttt{Meas}^{N\times I_0}$ and projections from $\texttt{Meas}^{N\times I_0}$ to $\texttt{Meas}$ are used, where I_0 and N are countable. Since the functor in Lemma 3 satisfies the conditions, there exists a final coalgebra for it, which by Lemma 4 is the universal type structure.

Suppose there are two final coalgebras (Ω, g) and (Ω', g'). There is a unique morphism from any coalgebra to (Ω, g) and to (Ω', g'), in particular the morphisms $f : \Omega \to \Omega'$, $f' : \Omega' \to \Omega$, $\mathrm{id}_\Omega : \Omega \to \Omega$ and $\mathrm{id}_{\Omega'} : \Omega' \to \Omega'$ are unique. Then $f' \circ f = \mathrm{id}_\Omega$ and $f \circ f' = \mathrm{id}_{\Omega'}$ by uniqueness, so $f = (f')^{-1}$ is an isomorphism. Any two final coalgebras are isomorphic.

By Lambek's Lemma, adapted as Theorem 2.3 in Viglizzo (2005), the morphism g in a final coalgebra is an isomorphism. □

Similarly to Lemma 3, it can be shown that type structures with unawareness of agents, unawareness of higher-order reasoning and three kinds of unawareness are coalgebras of the following functors V^a, V^b and V^c respectively.

$$V^a(M) = \left(S, \left(\bigsqcup_{i\in J'\subseteq J} \left(\Delta\left(M_0 \times \times_{j\in J'\setminus\{i\}} M_{J',j}\right) \times \Delta(M_{J',i})\right)\right)_{J\subseteq I}^{i\in J} \right)$$

$$V^b(M) = \left(S, \left(\bigsqcup_{n\leq k-1} \left(\Delta\left(M_0 \times \times_{j\in I\setminus\{i\}} M_{n,j}\right) \times \Delta(M_{n,i})\right)\right)_{1\leq k\leq\infty}^{i\in I} \right)$$

$$V^c(M) = \left(S_{\mathcal{F}}, \left(\bigsqcup_{\substack{\mathcal{E}\lhd\mathcal{F}\\ i\in J'\subseteq J}} \left(\Delta(M_{\mathcal{E},J',k,i}) \times \bigsqcup_{n\leq k-1} \Delta\left(M_{\mathcal{E},0} \times \times_{j\in J'\setminus\{i\}} M_{\mathcal{E},J',n,j}\right)\right)\right)_{1\leq k\leq\infty}^{i\in J\subseteq I} \right)_{\mathcal{F}\in\mathbb{F}}$$

Repeating the proof Lemma 4, it is clear that type morphisms for the different kinds of unawareness considered in Section 3 are coalgebra morphisms of the appropriate functors. The proof of Theorem 2 then follows along similar lines to Theorem 1.

Proof of Theorem 2. The functors V^a, V^b and V^c all satisfy the conditions of Theorem 7.1 of Viglizzo (2005), if the statement of that theorem is extended to countable products and disjoint unions. This can be done, keeping the same proof, since the operations are on measurable spaces. Therefore there exist final coalgebras for V^a, V^b and V^c, which are the universal type structures for unawareness of agents, unawareness of higher order reasoning, and three kinds of unawareness.

The uniqueness proof is the same as in Theorem 1. The morphisms in final coalgebras are isomorphisms, as argued in Theorem 1. □

Appendix B. Comparison to Heifetz, Meier, and Schipper (2011c)

Heifetz, Meier, and Schipper start from a complete lattice of spaces of primitive uncertainty, which corresponds to the present paper's lattice of σ-algebras on the space of primitive uncertainty. Both will be denoted $(S_{\mathcal{F}})_{\mathcal{F}\in\mathbb{F}}$ (the notation here differs from theirs in several ways). They set $Y^0_{\mathcal{F},i} = S_{\mathcal{F}}$ and define the first order beliefs of player i as $Q^1_{\mathcal{F},i} = \Delta(Y^0_{\mathcal{F},i})$, the space of compact regular Borel probability measures on $Y^0_{\mathcal{F},i}$ endowed with the topology of weak convergence. Inductively, the domain of $(k+1)$-order beliefs is $Y^k_{\mathcal{F},i} = S_{\mathcal{F}} \times \times_{j\neq i} \left(\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q^k_{\mathcal{E},j}\right)$ and the set of consistent $(k+1)$-order beliefs is

$$Q^{k+1}_{\mathcal{F},i} = \left\{(\mu^1,\ldots,\mu^{k+1}) \in Q^k_{\mathcal{F},i} \times \Delta\left(Y^k_{\mathcal{F},i}\right) : \mathrm{mrg}_{Y^{k-1}_{\mathcal{F},i}}\, \mu^{k+1} = \mu^k\right\}.$$

For each k, the function $\mathrm{mrg}_{Y^{k-1}_{\mathcal{F},i}} : \Delta\left(Y^k_{\mathcal{F},i}\right) \to \Delta\left(Y^{k-1}_{\mathcal{F},i}\right)$ is defined at the k-th induction step. To discuss the definition, some additional notation is needed. Denote a measurable subset of $Y^k_{\mathcal{F},i}$ by $E^{Y,k}_{\mathcal{F},i}$. It has the form $E_{\mathcal{F}} \times \times_{j\neq i} E^{Q,k}_{\mathcal{E}_j,j}$, where $\mathcal{E}_j \lhd \mathcal{F}$ for all j and $E^{Q,k}_{\mathcal{E}_j,j} = E^{Q,k-1}_{\mathcal{E}_j,j} \times E^{\Delta Y,k-1}_{\mathcal{E}_j,j} \subseteq Q^k_{\mathcal{F},i}$ is measurable and $E^{\Delta Y,k-1}_{\mathcal{E}_j,j} \subseteq \Delta\left(Y^{k-1}_{\mathcal{F},i}\right)$ is measurable.

The natural projection $p^k_{\mathcal{F},i} : Y^k_{\mathcal{F},i} \to Y^{k-1}_{\mathcal{F},i}$, extended from singleton elements of $Y^k_{\mathcal{F},i}$ to all measurable subsets of $Y^k_{\mathcal{F},i}$ by $p^k_{\mathcal{F},i}(E^{Y,k}_{\mathcal{F},i}) = \cup_{y\in E^{Y,k}_{\mathcal{F},i}} p^k_{\mathcal{F},i}(y)$, takes any measurable $E^{Y,k}_{\mathcal{F},i} = E_{\mathcal{F}} \times \times_{j\neq i}\left(E^{Q,k-1}_{\mathcal{E}_j,j} \times E^{\Delta Y,k-1}_{\mathcal{E}_j,j}\right)$ to the measurable $E^{Y,k-1}_{\mathcal{F},i} = E_{\mathcal{F}} \times \times_{j\neq i} E^{Q,k-1}_{\mathcal{E}_j,j} \subseteq Y^{k-1}_{\mathcal{F},i}$. The inverse of $p^k_{\mathcal{F},i}$, denoted $(p^k_{\mathcal{F},i})^{-1}$, takes a measurable $E_{\mathcal{F}} \times \times_{j\neq i} E^{Q,k-1}_{\mathcal{E}_j,j}$ to the measurable $E_{\mathcal{F}} \times \times_{j\neq i}\left(E^{Q,k-1}_{\mathcal{E}_j,j} \times \Delta\left(Y^k_{\mathcal{E}_j,j}\right)\right)$. The function $\mathrm{mrg}_{Y^{k-1}_{\mathcal{F},i}}$ can now be defined as $\left(\mathrm{mrg}_{Y^{k-1}_{\mathcal{F},i}}\, \mu^{k+1}\right)\left(E^{Y,k-1}_{\mathcal{F},i}\right) = \mu^{k+1}\left(\left(p^k_{\mathcal{F},i}\right)^{-1}\left(E^{Y,k-1}_{\mathcal{F},i}\right)\right)$.

In the limit, Heifetz, Meier, and Schipper define $Q_{\mathcal{F},i} = \left\{(\mu^1,\mu^2,\ldots) : (\mu^1,\ldots,\mu^k) \in Q^k_{\mathcal{F},i}, \forall k \geq 1\right\}$ and use a generalized Kolmogorov Extension Theorem to show that the belief function $\tau_{\mathcal{F},i} : Q_{\mathcal{F},i} \to \Delta\left(Y_{\mathcal{F},i}\right)$ is a homeomorphism, where $Y_{\mathcal{F},i} = S_{\mathcal{F}} \times \times_{j\neq i} \left(\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},j}\right)$.

The set of states of the world is $(Y_{\mathcal{F}})_{\mathcal{F}\in\mathbb{F}}$, where $Y_{\mathcal{F}} = S_{\mathcal{F}} \times \times_{i\in I} \left(\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},i}\right)$. Beliefs are extended from $Y_{\mathcal{F},i}$ to $Y_{\mathcal{F}}$ by imposing certainty of own belief (taking the Cartesian product with the Dirac δ-function on own type), similarly to Definition 1 of the present paper.

Starting from the same space of primitive uncertainty and the same set of agents and awareness levels, the set $(Y_{\mathcal{F}})_{\mathcal{F}\in\mathbb{F}}$ of Heifetz, Meier, and Schipper corresponds to $(\Omega_{\mathcal{F}})_{\mathcal{F}\in\mathbb{F}}$ in the present paper. Their type function τ_i maps each element of $Y_{\mathcal{F}}$ to $\Delta\left(Y_{\mathcal{E}}\right)$ for some $\mathcal{E} \lhd \mathcal{F}$, just like g_i maps $\Omega_{\mathcal{F}}$ to $\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta\left(\Omega_{\mathcal{E}}\right)$. Therefore the category of type structures with unawareness in Heifetz, Meier, and Schipper (2011c) differs from the present paper only in the use of topology.

In more detail, $Y_{\mathcal{F},i} = S_{\mathcal{F}} \times \times_{j\neq i} \Omega_{\mathcal{F},j} = \Omega_{\mathcal{F},-i}$ and $\Omega_{\mathcal{F},j} = \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},j}$. Also, $Q_{\mathcal{E},j}$ is homeomorphic to $\Delta\left(Y_{\mathcal{E},j}\right)$ via $\tau_{\mathcal{E},j}$. The relationship between Y, Ω and Q is (loosely) illustrated in the following diagram.

$$\begin{tikzcd} \Omega \arrow[rr, "\times"] & & Y \arrow[dl, "\Delta"] \\ & Q \arrow[ul, "\bigsqcup"] & \end{tikzcd}$$

If the functions τ_i and g_i were added to the triangular diagram, they would map an object to another object that is three steps ahead in the direction of the arrows. To show this, the triangle is unpacked in the diagram below.

$$\begin{tikzcd}
\bigsqcup_{\mathcal{E}\lhd\mathcal{F}} Q_{\mathcal{E},i} = \Omega_{\mathcal{F},i} \arrow[r, "\times"] \arrow[drr, dashed, "g^{-i}_{\mathcal{F},i}\ \tau_{\mathcal{F},i}"'] & Y_{\mathcal{F},i} = \times_{j\neq i}\Omega_{\mathcal{F},i} \arrow[r, "\Delta"] & \Delta\left(Y_{\mathcal{F},i}\right) \arrow[d, "\bigsqcup"] \\
Q_{\mathcal{F},i} \arrow[u, "\bigsqcup"] \arrow[urr, dashed] & & \bigsqcup_{\mathcal{E}\lhd\mathcal{F}} \Delta\left(\times_{j\neq i}\Omega_{\mathcal{E},j}\right)
\end{tikzcd}$$

Next, Heifetz, Meier, and Schipper (2007, 2011c) define projections between layers of the type structure and impose conditions relating projections to type functions. They then define events via projections. A similar exercise can be carried out in the context of the present paper.

In the universal type structure of this paper, for any $\mathcal{F} \in \mathbb{F}$ and $\mathcal{E} \lhd \mathcal{F}$, define a surjective projection operator $\rho^{\mathcal{F}}_{\mathcal{E}}$, with $\rho^{\mathcal{F}}_{\mathcal{F}}$ being the identity. Projections are assumed to commute. Events are of the form $\bigsqcup_{\mathcal{E}\rhd\mathcal{D}}(\rho^{\mathcal{E}}_{\mathcal{D}})^{-1}(E_{\mathcal{D}})$, where $E_{\mathcal{D}} \subseteq \Omega_{\mathcal{D}}$ is measurable. The negation $E^{c'}$ of E is nonstandard: $E^{c'} = \bigsqcup_{\mathcal{E}\rhd\mathcal{D}}(\rho^{\mathcal{E}}_{\mathcal{D}})^{-1}(\Omega_{\mathcal{D}} \setminus E_{\mathcal{D}})$. Conjunction of events is standard intersection. Disjunction is defined from conjunction and negation via de Morgan's laws and is nonstandard because of the nonstandard negation.

The same conditions on the interaction of projections and type functions in Heifetz, Meier, and Schipper (2007) can be imposed in this paper's type structure.

0) $g_i(t_{\mathcal{F}}) \in \Delta(\Omega_{\mathcal{E}})$ for some $\mathcal{E} \lhd \mathcal{F}$,
1) If $\mathcal{D} \lhd \mathcal{E} \lhd \mathcal{F}$, $t \in \Omega_{\mathcal{F}}$ and $g_i(t) \in \Delta(\Omega_{\mathcal{D}})$, then $g_i(\rho^{\mathcal{F}}_{\mathcal{E}}(t))(\cdot) = g_i(t)(\cdot)$,
2) If $\mathcal{D} \lhd \mathcal{E} \lhd \mathcal{F}$, $t \in \Omega_{\mathcal{F}}$ and $g_i(t) \in \Delta(\Omega_{\mathcal{E}})$, then $g_i(\rho^{\mathcal{F}}_{\mathcal{D}}(t))(\cdot) = g_i(t)\left((\rho^{\mathcal{E}}_{\mathcal{D}})^{-1}(\cdot)\right)$,
3) If $\mathcal{D} \lhd \mathcal{E} \lhd \mathcal{F}$, $t \in \Omega_{\mathcal{F}}$ and $g_i(\rho^{\mathcal{F}}_{\mathcal{E}}(t)) \in \Delta(\Omega_{\mathcal{D}})$, then $g_i(t) \in \Delta(\Omega_{\mathcal{E}'})$ for some $\mathcal{E}' \rhd \mathcal{D}$.

Conditions 1, 2 and 3 are illustrated in the following commutative diagrams. In the diagrams, $\Omega_{\mathcal{F}}|_{\mathcal{D}}$ denotes the subset of $\Omega_{\mathcal{F}}$ that g_i maps to $\Delta(\Omega_{\mathcal{D}})$. The function $\Delta(\rho^{\mathcal{E}}_{\mathcal{D}})$ maps $\mu_{\mathcal{E}} \in \Delta(\Omega_{\mathcal{E}})$ to $\mu_{\mathcal{D}} \in \Delta(\Omega_{\mathcal{D}})$ such that $\mu_{\mathcal{D}}(E) = \mu_{\mathcal{E}}((\rho^{\mathcal{E}}_{\mathcal{D}})^{-1}(E))$. The symbol $\rhd$ at the dotted arrow means $\mathcal{E}' \rhd \mathcal{D}$.

$$\begin{tikzcd}
\Omega_{\mathcal{F}}|_{\mathcal{D}} \arrow[d, "\rho^{\mathcal{F}}_{\mathcal{E}}"'] \arrow[dr, "g_{\mathcal{F},i}"] & \\
\Omega_{\mathcal{E}}|_{\mathcal{D}} \arrow[r, "g_{\mathcal{E},i}"] & \Delta(\Omega_{\mathcal{D}})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{\mathcal{F}}|_{\mathcal{E}} \arrow[r, "g_{\mathcal{F},i}"] \arrow[d, "\rho^{\mathcal{F}}_{\mathcal{D}}"'] & \Delta(\Omega_{\mathcal{E}}) \arrow[d, "\Delta(\rho^{\mathcal{E}}_{\mathcal{D}})"] \\
\Omega_{\mathcal{D}}|_{\mathcal{D}} \arrow[r, "g_{\mathcal{D},i}"] & \Delta(\Omega_{\mathcal{D}})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{\mathcal{F}}|_{\mathcal{E}'} \arrow[r, "g_{\mathcal{F},i}"] \arrow[d, "\rho^{\mathcal{F}}_{\mathcal{E}}"'] & \Delta(\Omega_{\mathcal{E}'}) \arrow[d, dotted, "\rhd"] \\
\Omega_{\mathcal{E}}|_{\mathcal{D}} \arrow[r, "g_{\mathcal{E},i}"] & \Delta(\Omega_{\mathcal{D}})
\end{tikzcd}$$

It is possible to first define negation, conjunction, belief and awareness operations on general subsets of the type set, construct events using these operators and then derive projection operators from events. The resulting projections will satisfy the above conditions 0–4.

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